Applications of Mean Value Theorems to the Theory of the Riemann Zeta Function

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Abstract

In the first half of these lectures we discuss mean value theorems for functions representable by Dirichlet series and sketch several applications to the distribution of zeros of the Riemann zeta function. These include the clustering of zeros about the critical line, Levinson's result that a third of the zeros are on the critical line, and a conditional result on the number of simple zeros. The second half focuses on mean values of Dirichlet polynomials, particularly "long" ones. We then show how these can be used to investigate the pair correlation of the zeros of the zeta function and to conjecture the sixth and eighth power moments of the zeta function on the critical line.

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1 What is a Mean Value Theorem?

By a mean value theorem we mean an estimate for the average of a function. When $F(s)$ has a convergent Dirichlet series expansion in some half–plane $\mathrm{Re}\, s > \sigma_0$ of the complex plane, we typically take the average over a vertical segment:

$$\int_0^T |F(\sigma + it)|^2 \, dt \qquad \text{or} \qquad \int_0^T F(\sigma + it) \, dt \, .$$

The path of integration here need *not* lie in this half–plane. For example, we would like to know the size of the integrals

$$I_k(\sigma, T) = \int_0^T |\zeta(\sigma + it)|^{2k} \, dt \, ,$$

for $\sigma \geq 1/2$ and k a positive integer. Here $F(s) = \zeta(s)^k$ and its Dirichlet series converges only for $\sigma > 1$.

There are many variations. For example, one can consider a discrete mean value

$$\sum_{r=1}^{R} |F(\sigma_r + it_r)|^2 \, ,$$

where the points $\sigma_r + it_r$ lie in $\mathbb{C}$. Or, one can estimate the mean value of a Dirichlet polynomial

$$F(s) = F_N(s) = \sum_{n=1}^{N} a_n n^{-s}$$

of "length" N.

2 Mean Values and Zeros

Mean value estimates are very useful for studying the zeros of the zeta function; this is one of the reasons so much effort has been expended on them. One link between means and zeros can be seen in Jensen's Formula from classical function theory.

Theorem 2.1. **(Jensen's Formula)** Let $f(z)$ be analytic for $|z| \leq R$ and suppose that $f(0) \neq 0$. If $r_1, r_2, \ldots, r_n$ are the moduli of all the zeros of $f(z)$ inside $|z| \leq R$, then

$$\log\left(\frac{|f(0)|R^n}{r_1 r_2 \cdots r_n}\right) = \frac{1}{2\pi} \int_0^{2\pi} \log |f(Re^{i\theta})| \, d\theta \, .$$

Here we see that the mean value of $\log |f(z)|$ around a circle is related to the distribution of the zeros of $f(z)$ inside that circle. There is an analogous result for rectangles, which is often more useful when working with Dirichlet series, namely, *Theorem* 2.2. **(Littlewood's Lemma)** Let $f(s)$ be analytic and nonzero on the rectangle $\mathcal{C}$ with vertices σ_0, σ_1, $\sigma_1 + iT$, and $\sigma_0 + iT$, where $\sigma_0 < \sigma_1$. Then

$$\begin{aligned} 2\pi \sum_{\rho \in \mathcal{C}} \mathrm{Dist}(\rho) = \int_0^T \log|f(\sigma_0 + it)|\, dt - \int_0^T \log|f(\sigma_1 + it)|\, dt \\ + \int_{\sigma_0}^{\sigma_1} \arg f(\sigma + iT)\, d\sigma - \int_{\sigma_0}^{\sigma_1} \arg f(\sigma)\, d\sigma\,, \end{aligned}$$

where the sum runs over the zeros ρ of $f(s)$ in $\mathcal{C}$ and "Dist(ρ)" is the distance from ρ to the left edge of the rectangle.

When we use Littlewood's Lemma below, only the first term on the right–hand side will be significant. In order not to be too technical, we will always use the result in the form

$$2\pi \sum_{\rho \in \mathcal{C}} \mathrm{Dist}(\rho) = \int_0^T \log|f(\sigma_0 + it)|\, dt + \mathcal{E},$$

where $\mathcal{E}$ is an error term that can be ignored and might be different on different occassions. The integral of the logarithm usually cannot be dealt with directly, so we often use the following trick:

$$\begin{aligned} \frac{1}{T}\int_0^T \log|f(\sigma + it)|\, dt &= \frac{1}{2T}\int_0^T \log(|f(\sigma + it)|^2)\, dt \\ &\leq \frac{1}{2}\log\left(\frac{1}{T}\int_0^T |f(\sigma + it)|^2\, dt\right), \end{aligned}$$

where the inequality follows from the arithmetic–geometric mean inequality. In this way we see a direct connection between the location of the zeros within a rectangle and the type of mean values we have been considering.

3 A Sample of Important Estimates

Let

$$I_k(\sigma, T) = \int_0^T |\zeta(\sigma + it)|^{2k}\, dt\,.$$

When $k = 1$ we know that for each fixed $\sigma > 1/2$

$$I_1(\sigma, T) \sim c(\sigma)\, T\,,$$

as $T \to \infty$, where $c(\sigma)$ is a know function of σ. In 1918 Hardy and Littlewood [HL] proved that when $\sigma = 1/2$,

$$I_1(1/2, T) \sim T \log T\,.$$

What can such estimates tell us about the zeta function? Comparing the result for σ greater than $1/2$ with that for $\sigma = 1/2$, we see that the zeta function tends to assume, on average, much larger values on the critical line than to the right of it. Since it also has many zeros on the critical line, we should expect the zeta function to behave rather erratically there.

The next higher moment was determined in 1926 by Ingham [I], who proved that

$$I_2(1/2,T) \sim \frac{T}{2\pi^2}\log^4 T\,.$$

Unfortunately, no asymptotic estimate has been proved for any k greater than 2. It is known that for positive rational k[1],

$$I_k(1/2,T) \gg T\log^{k^2} T$$

(see Ramachandra [R] and Heath–Brown [H-B]). This is also known to hold for all positive k if the Riemann Hypothesis is true (see Ramachandra [R]). We expect that

$$I_k(1/2,T) \sim c_k T\log^{k^2} T\,,$$

but a proof seems a long way off. J. B. Conrey and A. Ghosh (unpublished) suggested that

$$c_k = \frac{a_k g_k}{\Gamma(k^2+1)}\,,$$

where

$$a_k = \prod_p \left(\left(1-\frac{1}{p}\right)^{k^2}\sum_{r=0}^{\infty}\frac{d_k^2(p^r)}{p^r}\right)$$

and g_k is an integer. Only recently has anyone put forth a plausible value for g_k. J. B. Conrey and A. Ghosh [CG] conjectured that $g_3 = 42$, and J. B. Conrey and the author [CGO] conjecured that $g_4 = 24024$. Then J. Keating and N. Snaith [KS], using random matrix theory, conjectured a value for g_k for every complex number k with $\operatorname{Re} k > -1/2$. For integer values of k, their conjecture takes the form $g_k = (k^2!)\prod_{j=0}^{k-1} j!/(j+k)!\,.$

Another type of mean value important for applications is

$$\int_0^T \left|\zeta^{(j)}(\sigma+it)M_N(\sigma+it)\right|^2 dt\,, \tag{3.1}$$

where[2]

$$M_N(s) = \sum_{1\le n\le N}\frac{\mu(n)}{n^s}P\Big(\frac{\log n}{\log N}\Big)$$

[1] Editors' comment: See the Appendix of the lecture by D.W. Farmer, page 185, for a discussion of the $\gg$ notation.

[2] Editors' comment: The Möbius function, $\mu(n)$, is defined in the lectures of D.A. Goldston, page 79, equation 2.10.

and $P(x)$ is a polynomial. Since

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} \qquad (\operatorname{Re} s > 1)\,,$$

we can view $M_N(s)$ as an approximation to the reciprocal of $\zeta(s)$ in $\operatorname{Re} s > 1$. We might then expect the approximation to hold (in some sense) inside the critical strip as well. If that is the case, multiplying the zeta function by $M_N(s)$ should dampen (or mollify) the large values of zeta. Below we will see two applications of this idea. The most general estimates known for such integrals are due to Conrey, Ghosh, and the author [CGG2], who obtained asymptotic estimates for them when the length of the Dirichlet polynomial $M_N(s)$ is $N = T^\theta$ with $\theta < 1/2$. Later, Conrey [C] used Kloosterman sum techniques to show that these formulas also hold when $\theta < 4/7$.

We conclude this section by mentioning a few discrete mean value results. The author [G] proved asymptotic estimates for the sums

$$\sum_{0 \le \gamma \le T} |\zeta^{(j)}(\rho)|^2,$$

assuming the Riemann Hypothesis is true. Here γ runs over the ordinates of the zeros $\rho = 1/2 + i\gamma$ of $\zeta(s)$. Conrey, Ghosh, and Gonek [CGG2] proved discrete versions of the mollified mean values (3.1), namely

$$\sum_{0<\gamma<T} \left|\zeta^{'}(\rho) M_N(\rho)\right|^2,$$

under the assumption of the Riemann Hypothesis and the Generalized Lindelöf Hypothesis.

4 Application: A Simple Zero–Density Estimate

We want to show that there are relatively few zeros of the zeta function in the right half of the critical strip. Let $1/2 < \sigma_0 < 1$ be a fixed real number and let $\mathcal{C}$ be the rectangle in the complex plane with vertices at 2, $2+iT$, $\sigma_0 + iT$, σ_0. Applying our (simplified) version of Littlewood's Lemma, we see that

$$\sum_{\rho \in \mathcal{C}} \operatorname{Dist}(\rho) = \frac{1}{2\pi} \int_0^T \log(|\zeta(\sigma_0 + it)|)\, dt + \mathcal{E},$$

Where $\operatorname{Dist}(\rho)$ is the distance of the zero $\rho = \beta + i\gamma$ of the zeta function from the line $\operatorname{Re} s = \sigma_0$. Now let σ be a fixed real number with $\sigma_0 < \sigma < 1$ and write $N(\sigma, T)$ for the number of zeros with $\sigma < \beta \le 2$ and $0 < \gamma < T$. On the one hand, we have

$$\sum_{\rho \in \mathcal{C}} \mathrm{Dist}(\rho) \ge \sum_{\substack{\rho \in \mathcal{C} \\ \sigma \le \beta}} \mathrm{Dist}(\rho) \ge (\sigma - \sigma_0) N(\sigma, T).$$

On the other hand,

$$\begin{aligned} \frac{1}{2\pi} \int_0^T \log(|\zeta(\sigma_0 + it)|)\, dt &= \frac{1}{4\pi} \int_0^T \log(|\zeta(\sigma_0 + it)|^2)\, dt \\ &\le \frac{T}{4\pi} \log(\frac{1}{T} \int_0^T |\zeta(\sigma_0 + it)|^2)\, dt \end{aligned}$$

by the arithmetic–geometric mean inequality, as before. The integral on the last line is $I_k(\sigma_0, T)$, which we have seen is $\sim c(\sigma_0)T$, where $c(\sigma_0)$ is positive and independent of T. Thus, the last expression is $O(T)$. It follows that

$$N(\sigma, T) \ll T\,.$$

Since $N(T) \sim \frac{T}{2\pi} \log T$, we may interpret this as saying that the proportion of zeros to the right of any line $\mathrm{Re}\, s = \sigma > 1/2$ is infinitesimal.

This, the first zero–density estimate, was proved by H. Bohr and E. Landau [BL] in 1914. Since then there have been much stronger results, typically of the form

$$N(\sigma, T) \ll T^{\lambda(\sigma)}\,,$$

where $\lambda(\sigma) < 1$ for $\sigma > 1/2$. Nevertheless, the underlying idea in the proof of many of these results already appears here.

5 Application: Levinson's Method

Zero–density theorems tell us there are (relatively) few zeros to the right of the critical line. Our goal here is to sketch the method of Levinson [L], which shows that there are many zeros *on* it.

Recall that[3]

$$N(T) = \#\{\rho = \beta + i\gamma \mid \zeta(\rho) = 0, \quad 0 < \gamma < T\} \sim \frac{T}{2\pi} \log T$$

and let

$$N_0(T) = \#\,\{\rho = 1/2 + i\gamma \mid \zeta(\rho) = 0, \quad 0 < \gamma < T\}$$

denote the number of zeros on the critical line up to height T. The important estimations of $N_0(T)$ were:

[3] Editors' comment: See Section 7 of the lectures by D.R. Heath-Brown starting on page 1.

$N_0(T) \to \infty$	G. H. Hardy (1914)
$N_0(T) > cT$	G. H. Hardy-J. E. Littlewood (1921)
$N_0(T) > c'N(T)$	A. Selberg (1942)
$N_0(T) > \frac{1}{3}N(T)$	N. Levinson (1974)
$N_0(T) > \frac{2}{5}N(T)$	J. B. Conrey (1989)

Levinson's method begins with the following fact first proved by Speiser [Sp].

Theorem 5.1. **(Speiser)** The Riemann Hypothesis is equivalent to the assertion that $\zeta'(s)$ does not vanish in the left half of the critical strip.

In the early seventies, N. Levinson and H. L. Montgomery [LM] proved a quantitative version of this. Let

$$N'_{-}(T) = \#\{\rho' = \beta' + i\gamma' \mid \zeta'(\rho') = 0, -1 < \beta' < 1/2, \quad 0 < \gamma' < T\}$$

and

$$N_{-}(T) = \#\{\rho = \beta + i\gamma \mid \zeta(\rho) = 0, -1 < \beta < 1/2, \quad 0 < \gamma < T\} .$$

Theorem 5.2. **(Levinson-Montgomery)** We have $N_{-}(T) = N'_{-}(T) + O(\log T)$.

The idea behind the proof is as follows. Let $0 < a < 1/2$ and let $\mathcal{C}$ denote the positively oriented rectangle with vertices $a + iT/2$, $a + iT$, $-1 + iT$, and $-1 + iT/2$. It is not difficult to show that

$$\Delta \arg \frac{\zeta'}{\zeta}(s)\Big|_{\mathcal{C}} = O(\log T),$$

independently of a. Given this, we see that

$$2\pi(\#\text{ zeros of } \zeta'(s) \text{ in } \mathcal{C} - \#\text{ zeros of } \zeta(s) \text{ in } \mathcal{C}) = O(\log T).$$

The theorem now follows on observing that a was arbitrary, and by "adding" rectangles with top and bottom edges, respectively, at T and $T/2$, $T/2$ and $T/4, \ldots$.

We now sketch Levinson's method. We have just seen that $N_{-}(T) = N'_{-}(T) + O(\log T)$. Now, the nontrivial zeros of $\zeta(s)$ are symmetric about the critical line. Hence, the number of them lying to the right of the critical line up to height T is also $N_{-}(T)$. Therefore

$$\begin{aligned} N(T) &= N_0(T) + 2N_{-}(T) \\ &= N_0(T) + 2N'_{-}(T) + O(\log T), \end{aligned}$$

or

$$N_0(T) = N(T) - 2N'_{-}(T) + O(\log T).$$

The size of the first term on the right–hand side of the last line is known, namely, $(1+o(1))\frac{T}{2\pi}\log T$. Hence, if we can determine a sufficiently small upper bound for $N'_-(T)$, we can deduce a lower bound for $N_0(T)$.

To find such an upper bound it is convenient to first note that the zeros of $\zeta'(s)$ in the region $-1<\sigma<1/2,\ 0<t<T$, are identical to the zeros of $\zeta'(1-s)$ in the reflected region $1/2<\sigma<2,\ 0<t<T$. One can also show, by the functional equation for the zeta function, that $\zeta'(1-s)$ and $G(s)=\zeta(s)+\zeta'(s)/L(s)$ have the same zeros in $1/2<\sigma<2,\ 0<t<T$, where $L(s)$ is essentially $\frac{1}{2\pi}\log T$. It turns out to be technically advantageous to count the zeros of $G(s)$ rather than those of $\zeta'(1-s)$.

To bound the number of zeros of $G(s)$ in this region, we apply Littlewood's Lemma. Let $a=1/2-\delta/\log T$, with δ a small positive number, and let $\mathcal{R}_a$ denote the rectangle whose vertices are at $a, 2, 2+iT$, and $a+iT$. It would be natural to apply our abbreviated form of the lemma to obtain

$$\sum_{\rho^*\in\mathcal{R}_a}\mathrm{Dist}(\rho^*)=\frac{1}{2\pi}\int_0^T\log|G(a+it)|dt+\mathcal{E},$$

where ρ^* denotes a zero of $G(s)$ and $\mathrm{Dist}(\rho^*)$ is its distance to the left edge of $\mathcal{R}_a$. However, in the next step, when we apply the arithmetic–geometric mean inequality to the integral, we would lose too much. To avoid this loss, we first mollify $G(s)$ and then apply Littlewood's Lemma in the form

$$\sum_{\substack{\rho^{**}\in\mathcal{R}_a\\ GM(\rho^{**})=0}}\mathrm{Dist}(\rho^{**})=\frac{1}{2\pi}\int_0^T\log|G(a+it)M(a+it)|dt+\mathcal{E}\,.$$

Here $M(s)=\sum_{n\le T^\theta}\ a_n/n^s$, with $a_n=\mu(n)n^{a-1/2}\left(1-\frac{\log n}{\log T^\theta}\right)$ and $\theta>0$, approximates $1/\zeta(s)$. Note that among the zeros of $G(s)M(s)$ in $\mathcal{R}_a$ are all the zeros of $G(s)$ in $\mathcal{R}_a$. Therefore we have

$$\begin{aligned}\sum_{\substack{\rho^{**}\in\mathcal{R}_a\\ GM(\rho^{**})=0}}\mathrm{Dist}(\rho^{**})\ge&\sum_{\substack{\rho^{*}\in\mathcal{R}_a\\ G(\rho^{*})=0}}\mathrm{Dist}(\rho^{*})\\ \ge&\sum_{\substack{\rho^{*}\in\mathcal{R}_a,\mathrm{Re}\rho^*>1/2\\ G(\rho^{*})=0}}\mathrm{Dist}(\rho^{*})\\ \ge&(1/2-a)N'_-(T)\,.\end{aligned}$$

We now see that

$$\begin{aligned}(1/2-a)N'(T) &\le \frac{1}{2\pi}\int_0^T \log|GM(a+it)|dt + \mathcal{E}\\ &= \frac{1}{4\pi}\int_0^T \log|GM(a+it)|^2 dt + \mathcal{E}\\ &\le \frac{T}{4\pi}\log\left(\frac{1}{T}\int_0^T |GM(a+it)|^2 dt\right) + \mathcal{E}\,.\end{aligned}$$

Thus, we require an estimate for

$$\int_0^T |GM(a+it)|^2 dt\,.$$

This is similar to a mean value we saw in Section 3. Levinson was able prove an asymptotic estimate for this integral when $\theta = 1/2-\epsilon$ with ϵ arbitrarily small. The resulting upper bound for $N'_{-}(T)$ then led to the lower bound

$$N_0(T) > (1/3 + o(1))\, N(T).$$

Conrey later proved an asymptotic estimate for the integral when $\theta = 4/7-\epsilon$. This led to

$$N_0(T) > (2/5 + o(1))\, N(T).$$

The form of the asymptotic estimate in both cases is the same as a function of θ, and D. Farmer [F] has given heuristic arguments to suggest that this remains the case even when θ is arbitrarily large. From Farmer's conjecture it follows that

$$N_0(T) \sim N(T)\ .$$

Before concluding this section, we remark that had we introduced a mollifier into our proof of the Bohr–Landau result in the previous section, we would have obtained a much stronger zero–density estimate of the form we alluded to previously, namely $N(\sigma,T) \ll T^{\lambda(\sigma)}$, with $\lambda(\sigma) < 1$.

6 Application: The Number of Simple Zeros

Our third application demonstrates the use of discrete mean value theorems.

Let

$$N_s(T) = \#\{\rho = \beta + i\gamma \mid \zeta(\rho) = 0, \zeta'(\rho) \neq 0, \quad 0 < \gamma < T\}$$

denote the number of simple zeros of the zeta function in the critical strip with ordinates between 0 and T. It is believed that all the nontrivial zeros are on the critical line and simple, in other words, that $N(T) = N_0(T) = N_s(T)$ for every $T > 0$. Unconditionally, it is known that at least 2/5 of the zeros are simple (see Conrey [C]). In 1973, H. Montgomery [M], used his pair correlation method to show that if the Riemann Hypothesis is true, then more than 2/3 of the zeros are simple. In other words,

$$N_s(T) > 2/3N(T)$$

provided that T is sufficiently large. We will outline his argument in section 8. Here we briefly describe a different method of Conrey, Ghosh, and Gonek [CGG1], which shows that on the stronger hypotheses of RH and the Generalized Lindelöf Hypothesis, one can replace the 2/3 above by $19/27 = .703\ldots$.

By the Cauchy–Schwarz inequality, we have

$$\Big|\sum_{0<\gamma<T} \zeta^{'}(1/2+i\gamma)M_N(1/2+i\gamma)\Big|^2 \le \Big(\sum_{\substack{0<\gamma\le T\\ 1/2+i\gamma \text{ is simple}}} 1\Big)\Big(\sum_{0<\gamma<T} |\zeta^{'}(\rho)M_N(\rho)|^2\Big),$$

where $M_N(s)$ is a Dirichlet polynomial of length N with coefficients similar, but not identical, to those of $M(s)$ in the last section. Its purpose is also similar: to mollify $\zeta^{'}(1/2+i\gamma)$ so as to minimize the loss in applyng the Cauchy–Schwarz inequality. If one assumes RH, the sum on the left–hand side is easy to compute and turns out to be $\sim 19/24N(T)\log T$. The sum on the right–hand side is much more difficult to treat, but one can show that if RH and GLH are true, then it is $\sim 57/64N(T)\log^2 T$. Inserting these estimates into the inequality above and solving for $N_s(T)$, we obtain the stated result. An elaboration of the method leads to the conclusion that, on the same hypotheses, over 95.5% of the zeros of $\zeta(s)$ are either simple or double.

7 Mean Values of Dirichlet Polynomials

From now on we will focus on mean values of Dirichlet polynomials. Let

$$A(s) = A_N(s) = \sum_{n=1}^{N} a_n n^{-s}$$

be a Dirichlet polynomial of length N and let $s = \sigma + it$. The Classical Mean Value Therorem for Dirichlet polynomials is

Theorem 7.1. **(Classical Mean Value Theorem)**

$$\int_0^T |\sum_{n=1}^{N} a_n n^{-s}|^2 dt = \Big(T + O(N\log N)\Big)\sum_{n=1}^{N} |a_n|^2\, n^{-2\sigma}\,.$$

A more precise version due to H. L. Montgomery and R. C. Vaughan is

Theorem 7.2. **(Montgomery–Vaughan)**

$$\int_0^T |\sum_{n=1}^{N} a_n n^{-s}|^2 dt = \sum_{n=1}^{N} |a_n|^2\, n^{-2\sigma} \Big(T + O(n)\Big)\,.$$

From this we see that if $N = o(T)$, then

$$\int_0^T |\sum_{n=1}^{N} a_n n^{-\sigma - it}|^2 dt \sim\ T \sum_{n=1}^{N} |a_n|^2 n^{-2\sigma}\,.$$

On the other hand, if $N \gg T$ the O–term dominates and we have only

$$\int_0^T |\sum_{n=1}^{N} a_n n^{-\sigma - it}|^2 dt \ll\ N \sum_{n=1}^{N} |a_n|^2 n^{-2\sigma}\,.$$

It is natural to ask whether this is the actual size of the mean when N is larger than T. The following example answers this question.

Example. Let each $a_n = 1$ and take $\sigma = 1/2$. Montgomery and Vaughan's mean value formula gives

$$\begin{aligned}\int_0^T |\sum_{n=1}^{N} n^{-\frac{1}{2} - it}|^2 dt =& \sum_{n=1}^{N} \frac{1}{n}\Big(T + O(n)\Big)\\ =& T\,(\log N + O(1)) + O(N)\\ =& \begin{cases} (1 + o(1)) T \log N & \text{if} \quad N = O(T)\ , \\ O(N) & \text{if} \quad N > T^{\alpha}\ (\alpha > 1)\ . \end{cases}\end{aligned}$$

We can also evaluate this using a crude form of the approximate functional equation for the zeta function (see Titchmarsh [T], p.77), namely

$$\zeta(s) = \sum_{1 \le n \le N} n^{-s} + \frac{N^{1-s}}{s-1} + O(N^{-\sigma})\,.$$

Taking $\sigma = 1/2$, we obtain

$$\int_0^T \left|\sum_{n=1}^{N} n^{-1/2 - it}\right|^2 dt = \int_0^T \left|\zeta(1/2 + it) + \frac{N^{1/2 - it}}{1/2 - it} + O(N^{-1/2})\right|^2 dt\,.$$

Now, we know (see Titchmarsh [T]) that

$$\int_0^T \left|\zeta(1/2 + it)\right|^2 dt \sim T \log T\,.$$

Furthermore, it is easy to see that

$$\int_0^T \left|\frac{N^{1/2-it}}{1/2-it}\right|^2 dt = N\int_0^T \frac{1}{1/4+t^2}dt \sim \pi N \qquad \text{and} \qquad \int_0^T N^{-1}dt = T/N\,.$$

Hence, we find that

$$\int_0^T \big|\sum_{n=1}^N n^{-1/2-it}\big|^2 dt \sim \pi N$$

if $N \gg T^\alpha$ and $\alpha > 1$. Therefore, the O– term cannot be reduced in this case and, in fact, we can extract a new main term. So it makes sense to ask whether there is a useful general asymptotic formula for

$$\int_0^T |\sum_{n=1}^N a_n n^{-s}|^2 dt$$

when $N = T^\alpha$, $\alpha > 1$. In order to answer this, let us consider the proof of the Classical Mean Value Theorem. Squaring out and integrating, we obtain

$$\begin{aligned}\int_0^T \big|\sum_{n=1}^N a_n n^{-\sigma-it}\big|^2 dt =& \sum_{n=1}^N\sum_{m=1}^N \frac{a_n\overline{a}_m}{(nm)^\sigma}\int_0^T (m/n)^{it}\,dt\\ =& T\sum_{n=1}^N \frac{|a_n|^2}{n^{2\sigma}} + \sum_{\substack{1\le m,n\le N\\ m\ne n}} \frac{a_n\overline{a}_m}{(nm)^\sigma}\left(\frac{e^{iT\log(m/n)}-1}{i\log(m/n)}\right).\end{aligned}$$

The second sum consists of "off–diagonal" terms and is

$$\begin{aligned}&\ll \sum_{\substack{1\le m,n\le N\\ m\ne n}} \frac{|a_n a_m|}{(nm)^\sigma|\log(m/n)|} \le \frac{1}{2}\sum_{\substack{1\le m,n\le N\\ m\ne n}}\left(\frac{|a_n|^2}{n^{2\sigma}}+\frac{|a_m|^2}{m^{2\sigma}}\right)\frac{1}{|\log(m/n)|}\\ &= \sum_{\substack{1\le m,n\le N\\ m\ne n}} \frac{|a_n|^2}{n^{2\sigma}|\log(m/n)|} = \sum_{1\le n\le N}\frac{|a_n|^2}{n^{2\sigma}}\Bigg(\sum_{\substack{1\le m\le N\\ m\ne n}}\frac{1}{|\log(m/n)|}\Bigg).\end{aligned}$$

The inner sum is

$$\begin{aligned}&\ll \Bigg(\sum_{\substack{m\le N\\ |m-n|\le n/2}} + \sum_{\substack{m\le N\\ n/2<|m-n|}}\Bigg)\frac{1}{|\log(m/n)|}\\ &\ll \sum_{1\le h\le n/2}\frac{1}{|\log((n\pm h)/n)|} \quad + \sum_{\substack{m<n/2\ or\\ 3n/2<m\le N}}\frac{1}{|\log(m/n)|}\\ &\ll \sum_{1\le h\le N}\frac{n}{h} + \sum_{1\le m\le N} 1 \ll N\log N + N \ll N\log N\,.\end{aligned}$$

Hence, the off–diagonal terms are

$$\ll N \log N \sum_{1 \le n \le N} \frac{|a_n|^2}{n^{2\sigma}} .$$

We therefore find that

$$\int_0^T \Big| \sum_{n=1}^{N} a_n n^{-\sigma - it} \Big|^2 dt = \big(T + O(N \log N)\big) \sum_{n=1}^{N} \frac{|a_n|^2}{n^{2\sigma}} .$$

From this it is clear that if we want a precise formula when N is much larger than T, we need to estimate the off–diagonal terms more carefully.

Returning to our initial expression for these terms, we see that

$$\begin{aligned}
&\sum_{\substack{1 \le m,n \le N \\ m \ne n}} \frac{a_n \overline{a}_m}{(nm)^\sigma} \left(\frac{e^{iT \log(m/n)} - 1}{i \log(m/n)} \right) \\
&= 2\mathrm{Re} \sum_{1 \le n < m \le N} \frac{a_n \overline{a}_m}{(nm)^\sigma} \left(\frac{e^{iT \log(m/n)} - 1}{i \log(m/n)} \right) \\
&= 2\mathrm{Re} \sum_{1 \le n < N} \sum_{1 \le h \le N-n} \frac{a_n \overline{a}_{n+h}}{(n(n+h))^\sigma} \left(\frac{e^{iT \log((n+h)/n)} - 1}{i \log((n+h)/n)} \right) \\
&= 2\mathrm{Re} \sum_{1 \le h < N} \sum_{1 \le n \le N-h} \frac{a_n \overline{a}_{n+h}}{n^{2\sigma}} (1 + h/n)^{-\sigma} \left(\frac{e^{iT \log(1+h/n)} - 1}{i \log(1+h/n)} \right) .
\end{aligned}$$

For the sake of simplicity, consider only the terms with $h/n < 1/2$. In these $\log(1 + h/n)$ is approximately h/n and $(1 + h/n)^\sigma$ is approximately 1. These terms therefore contribute about

$$2\mathrm{Re} \sum_{1 \le h < N} \sum_{2h < n \le N-h} \frac{a_n \overline{a}_{n+h}}{n^{2\sigma - 1}} \left(\frac{e^{iTh/n} - 1}{ih} \right) .$$

To simplify further, we restrict our attention to the terms with $Th < n/2$. In these $(e^{iTh/n} - 1)/ih$ is approximately T/n, so their contribution is about

$$2T \, Re \sum_{h \ne 0} \sum_{n} a_n \overline{a}_{n+h} n^{-2\sigma} .$$

We can clearly estimate this if we have good estimates for the sums

$$\sum_{n=1}^{N} a_n \, \overline{a}_{n+h} .$$

In fact, this would be sufficient to estimate the terms we ignored as well. To state the final result obtained, we assume the a_n satisfy the following conditions (see Goldston and Gonek [GG] for the details):

1. (Normalization)

$$a_n \ll n^{\epsilon}.$$

2. There is a function $M(x)$ and a real number θ with $0 < \theta < 1$ such that

$$\sum_{n \le x} a_n = M(x) + O(x^{\theta}),$$

$M'(x) \ll x^{\epsilon}$, and $M''(x) \ll x^{-1+\epsilon}$.

3. There is a function $M(x,h)$, real numbers ϕ and η with $0 < \phi, \eta < 1$, such that

$$\sum_{n \le x} a_n \overline{a}_{n+h} = M(x,h) + O(x^{\phi})$$

uniformly for $h \le x^{\eta}$, and $M'(x,h) \ll (hx)^{\epsilon}$.

In applications it is often more convenient to estimate

$$\int_0^T \Big| \sum_{n=1}^{N} a_n n^{-s} - \int_1^N M'(x) x^{-s} dx \Big|^2 dt$$

rather than

$$\int_0^T \Big| \sum_{n=1}^{N} a_n n^{-s} \Big|^2 dt\,.$$

Here the integral involving $M'(x)$ may be thought of as an expected value. Also, it is much easier to work with a weighted mean

$$\int_{-\infty}^{\infty} \Psi_U\Big(\frac{t}{T}\Big) \Big| \sum_{n=1}^{N} a_n n^{-s} - \int_1^N M'(x) x^{-s} dx \Big|^2 dt\,,$$

where $\Psi_U(x)$ is nonnegative, has support in $[1-U^{-1}, 1+U^{-1}]$ with $U = \log^A T$, and satisfies $\Psi_U^{(j)}(x) \ll \log^j T$ and $\Psi_U(x) = 1$ in $[1+U^{-1}, 1-U^{-1}]$. It follows that the Fourier transform $\widehat{\Psi}_U(v)$ is approximately 1 for $|v| \le 1$ and drops off rapidly as $|v|$ increases. Thus $\Psi_U(t/T)$ is a smooth approximation to the characteristic function of the interval $[0,T]$. Our result is

Theorem 7.3. **(Goldston–Gonek)** Let $\epsilon > 0$ be arbitraily small, $\sigma < 1$, and

θ, ϕ, η as above. Then for $T \ll N \ll T^{(1-\epsilon)/(1-\eta)}$ we have

$$\int_{-\infty}^{\infty} \Psi(\frac{t}{T}) \Big| \sum_{n=1}^{N} a_n n^{-s} - \int_{1}^{N} M'(x) x^{-s} dx \Big|^2 dt$$
$$= \widehat{\Psi}(0) T \sum_{n \le N} |a_n|^2 n^{-2\sigma}$$
$$+ \ 4\pi (\frac{T}{2\pi})^{2-2\sigma} \mathrm{Re} \int_{T/2\pi N}^{\infty} \Bigg(\sum_{1 \le h \le 2\pi N v/T} M'(\frac{hT}{2\pi v}, h) h^{1-2\sigma} \Bigg) \frac{\widehat{\Psi}(v)}{v^{2-2\sigma}} \, dv$$
$$- \ 4\pi (\frac{T}{2\pi})^{2-2\sigma} \mathrm{Re} \int_{T^{\epsilon}/2\pi N}^{\infty} \Bigg(\int_{0}^{2\pi N v/T} \big| M'(\frac{uT}{2\pi v}) \big|^2 u^{1-2\sigma} du \Bigg) \frac{\widehat{\Psi}(v)}{v^{2-2\sigma}} \, dv$$
$$+ \ O(N^{1-2\sigma+max(\theta,\phi)+5\epsilon}) + O(N^{2-2\sigma+5\epsilon} T^{-1}) + O(N^{2\epsilon}) \, .$$

A similar formula can be proved for the tails of Dirichlet series minus their expected value, that is, for $\sum_{n>N} a_n n^{-s} - \int_N^{\infty} M'(x) x^{-s} dx$. One can also estimate the "mixed" means

$$\int_{-\infty}^{\infty} \Psi(\frac{t}{T}) \Bigg(\sum_{n=1}^{N} a_n n^{-s} - \int_{1}^{N} M_a'(x) x^{-s} dx \Bigg) \overline{\Bigg(\sum_{n=1}^{N} b_n n^{-s} - \int_{1}^{N} M_b'(x) x^{-s} dx \Bigg)} dt \, ,$$

where $M_a'(x)$ and $M_b'(x)$ have an obvious meaning. Finally, one can show that the integrals of "crossed" expressions consisting of a Dirichlet polynomial times the complex conjugate of the tail of a Dirichlet series (minus their expected values in both cases) are generally of smaller order than means involving a polynomial times a polynomial or a tail times a tail.

We now turn to applications of long mean value theorems.

8 Application: A Lower Bound for $F(\alpha)$

H. L. Montgomery [M] studied the function[4]

$$F(\alpha) = (\frac{T}{2\pi} \log T)^{-1} \sum_{0<\gamma,\gamma' \le T} T^{i\alpha(\gamma-\gamma')} \frac{4}{4+(\gamma-\gamma')^2} \, .$$

It is known that $F(\alpha)$ is even and nonnegative, and Montgomery showed that if the Riemann Hypothesis is true, then

$$F(\alpha) = (1+o(1)) T^{-2\alpha} \log T + \alpha + o(1) \tag{8.1}$$

for $|\alpha| \le 1$. He also conjectured that

$$F(\alpha) = (1+o(1)) \tag{8.2}$$

[4]Editors' comment: The form factor, $F(\alpha)$, is also discussed in Sections 4 and 6 of the lectures of D.A. Goldston, page 79.

when $1 \le |\alpha| \le A$ with A arbitrarily large. The only known nontrivial lower bound for $F(\alpha)$ when $|\alpha| \ge 1$ is given by

Theorem 8.1. **(Goldston–Gonek–Ozluk–Snyder)** Assume the Generalized Riemann Hypothesis. Then

$$F(\alpha) \ge 3/2 - |\alpha| - \epsilon$$

uniformly for $1 \le |\alpha| \le 3/2 - 2\epsilon$ and $T \ge T_0(\epsilon)$.

See [GGOS].

Sketch of the proof. First we sketch the derivation of Montgomery's results (8.1) and (8.2).

We begin with the explicit formula

$$\begin{aligned}-2\sum_{0<\gamma\le T} x^{i(\gamma-t)}\,\frac{1}{1+(t-\gamma)^2} =& x^{-1}\left(\sum_{n\le x}\Lambda(n)n^{1/2-it} - \int_1^x u^{1/2-it}du\right)\\ &+ x\left(\sum_{n>x}\Lambda(n)n^{-3/2-it} - \int_x^\infty u^{-3/2-it}du\right) + \mathcal{E}\,,\end{aligned}$$

where $\mathcal{E}$, as usual, denotes an ignorable error term. Integrating the modulus squared of both sides (see Montgomery [M] for details), we see that the left–hand side is

$$\begin{aligned}\int_0^T \left|2\sum_{0<\gamma\le T} x^{i(\gamma-t)}\,\frac{1}{1+(t-\gamma)^2}\right|^2 dt &= 2\pi \sum_{0<\gamma,\gamma'\le T} x^{i(\gamma-\gamma')}\,\frac{4}{4+(\gamma-\gamma')^2} + \mathcal{E}\\ &= 2\pi\, F(x,T) + \mathcal{E}\,,\end{aligned}$$

where we write

$$F(x,T) = \sum_{0<\gamma,\gamma'\le T} x^{i(\gamma-\gamma')}\,\frac{4}{4+(\gamma-\gamma')^2}\,.$$

Note that

$$F(\alpha) = (\frac{T}{2\pi}\log T)^{-1} F(T^\alpha, T)\,.$$

Equating this with the mean squared modulus of the right–hand side, we find that

$$\begin{aligned}2\pi\, F(x,T) = \int_0^T \Bigg| & x^{-1}\left(\sum_{n\le x}\Lambda(n)n^{\frac12-it} - \int_1^x u^{\frac12-it}du\right)\\ &+ x\left(\sum_{n>x}\Lambda(n)n^{-\frac32-it} - \int_x^\infty u^{-\frac32-it}du\right)\Bigg|^2 dt + \mathcal{E}\,.\end{aligned} \tag{8.3}$$

Case 1. $x = T^\alpha$, $\alpha < 1$. Applying the Montgomery–Vaughan mean value theorem in a straightforward way, we obtain (8.1).

Case 2. $x = T^\alpha$, $1 \le \alpha < A$. Applying Theorem 7.3 and a strong form of the Twin Prime Conjecture, we obtain the conjecture 8.2. More precisely, if we assume that

$$\sum_{n \le y} \Lambda(n)\Lambda(n+h)) = c(h)x + O(x^{1/2+\epsilon})\,,$$

where $c(h)$ is defined in Theorem 8.2 below, we obtain 8.2 with $A = 2$. If we also assume there is significant cancellation among the O–terms when averaged over h, we obtain (8.2) with A arbitrarily large.

Theorem 8.1 is proved as follows. We have no proof of the Twin Prime Conjecture, but we have its analogue for the functions

$$\lambda_Q(n) = \sum_{q \le Q} \frac{\mu^2(q)}{\phi(q)} \sum_{\substack{d|q \\ d|n}} d\mu(d)\ ,$$

which approximate the $\Lambda(n)$'s. Let us rewrite (8.3) as

$$2\pi\, F(x,T) = \int_0^T \left|\mathbb{A}(x,t) + \mathbb{A}^*(x,t)\right|^2 dt + \mathcal{E}\,.$$

Also, let $\mathbb{A}_Q(x,t)$ and $\mathbb{A}^*_Q(x,t)$ be the same as $\mathbb{A}(x,t)$ and $\mathbb{A}^*(x,t)$, respectively, but with the $\Lambda(n)$'s replaced by $\lambda_Q(n)$'s. Clearly we have

$$0 \le \int_{-\infty}^{\infty} \Psi_U(\frac{t}{T}) \left| \big(\mathbb{A}(x,t) + \mathbb{A}^*(x,t)\big) - \big(\mathbb{A}_Q(x,t) + \mathbb{A}^*_Q(x,t)\big) \right|^2 dt\,.$$

It follows that

$$\begin{aligned} 2\mathrm{Re} \int_{-\infty}^{\infty} & \Psi_U(\frac{t}{T}) \left(\mathbb{A}\overline{\mathbb{A}_Q} + \mathbb{A}^*\overline{\mathbb{A}^*_Q} + \mathbb{A}\overline{\mathbb{A}^*_Q} + \mathbb{A}^*\overline{\mathbb{A}_Q} - \mathbb{A}_Q\overline{\mathbb{A}^*_Q} \right) dt \\ & - \int_{-\infty}^{\infty} \Psi_U(\frac{t}{T}) \big(\mathbb{A}_Q\overline{\mathbb{A}_Q} + \mathbb{A}^*_Q\overline{\mathbb{A}^*_Q}\big) dt \\ & \le \int_{-\infty}^{\infty} \Psi_U(\frac{t}{T}) \left|\big(\mathbb{A}(x,t) + \mathbb{A}^*(x,t)\big)\right|^2 dt \\ & = 2\pi\, F(x,T)\,. \end{aligned}$$

The coefficient correlation sums needed to estimate the long mean values here are

$$\sum_{n \le y} \Lambda(n)\lambda_Q(n+h) \qquad \text{for} \qquad \mathbb{A}\overline{\mathbb{A}_Q} \quad \text{and} \quad \mathbb{A}^*\overline{\mathbb{A}^*_Q}$$

and

$$\sum_{n \le y} \lambda_Q(n)\lambda_Q(n+h) \qquad \text{for} \qquad \mathbb{A}_Q\overline{\mathbb{A}_Q} \quad \text{and} \quad \mathbb{A}^*_Q\overline{\mathbb{A}^*_Q}\ .$$

These are available from

Theorem 8.2. **(J. Friedlander–D. Goldston)** Assume the Generalized Riemann Hypothesis. Let $Q = y^\delta$ with $1/4 \le \delta \le 1/2$. Then

$$\sum_{n \le y} \Lambda(n)\lambda_Q(n+h) \qquad \text{and} \qquad \sum_{n \le y} \lambda_Q(n)\lambda_Q(n+h)$$

are both $= c(h)y + O(y^{\frac{1}{2}+\delta+\epsilon})$ uniformly for $1 \le h \le y^{1-\epsilon}$, where

$$c(h) = \begin{cases} 2\prod_{p>2}\left(1 - \frac{1}{(p-1)^2}\right)\prod_{\substack{p>2 \\ p \mid h}}\left(\frac{p-1}{p-2}\right) & \text{if h is even ,} \\ 0 & \text{if h is odd .} \end{cases}$$

Applying this to our terms, taking $X = T^\alpha$, and choosing δ optimally as a function of α leads to

$$F(\alpha, T) \ge \frac{3}{2} - |\alpha| - \epsilon$$

for $1 \le \alpha \le 3/2 - 2\epsilon$.

9 Application: The 6th and 8th Power Moments of the Zeta Function

In Section 3 we defined

$$I_k(1/2, T) = \int_0^T |\zeta(1/2 + it)|^{2k}\, dt$$

for positive values of k. Recall that Hardy and Littlewood showed that

$$\int_0^T |\zeta(1/2 + it)|^2\, dt \sim T \log T\,,$$

Ingham showed that

$$\int_0^T |\zeta(1/2 + it)|^4\, dt \sim \frac{1}{2\pi^2} T \log^4 T\,,$$

and no other asymptotic formula has ever been proved. In the mid 1990's J. B. Conrey and A. Ghosh [CG] made the following

Conjecture 1. **(Conrey–Ghosh)** As $T \to \infty$,

$$\int_0^T |\zeta(1/2 + it)|^6\, dt \sim \frac{42}{9!}\prod_p \left(\sum_{r=0}^{\infty} \frac{d_3(p^r)^2}{p^r}\right) T \log^9 T\,,$$

where $d_3(n)$ denotes the number of ways to write n as a product of three positive integers.

J. B. Conrey and I [CGO] followed this a few years later with

Conjecture 2. **(Conrey–Gonek)** As $T \to \infty$,

$$\int_0^T |\zeta(1/2+it)|^8 \, dt \sim \frac{24024}{16!} \prod_p \left(\sum_{r=0}^{\infty} \frac{d_4(p^r)^2}{p^r} \right) T \log^{16} T \,,$$

where $d_4(n)$ is the four–fold divisor function.

All these results and conjectures relied on estimating mean values of Dirichlet polynomial approximations to powers of the zeta function. It should be mentioned that the Keating–Snaith [KS] conjecture previously refered to used an entirely different method, namely, they modeled the zeta function by characteristic polynomials of random unitary matrices.

Here we sketch our method for the 6th and 8th moment conjectures. It gives the 2nd and 4th moment asymptotics as well. We begin with a discussion of the approximate functional equation.

For $s = \sigma + it$ and $\sigma > 1$, $\zeta^k(s)$ has the Dirichlet series expansion

$$\zeta^k(s) = \prod_p \left(1 - p^{-s}\right)^{-k} = \prod_p \left(1 + \frac{d_k(p)}{p^s} + \frac{d_k(p^2)}{p^{2s}} + \cdots \right) = \sum_{n=1}^{\infty} \frac{d_k(n)}{n^s} \,,$$

where $d_k(p^j) = (-1)^j \binom{-k}{j}$ is the kth divisor function. The series does not converge when $\sigma \le 1$, but we can approximate $\zeta^k(s)$ in this region by an expression of the form

$$\zeta(s)^k = \sum_{n=1}^{N} \frac{d_k(n)}{n^s} + \chi(s)^k \sum_{n=1}^{M} \frac{d_k(n)}{n^{1-s}} + \mathcal{E}_k(s) \,,$$

where $\mathcal{E}_k(s)$ denotes an error term. This is an approximate functional equation. We write it as

$$\zeta(s)^k = \mathcal{D}_{k,N}(s) + \chi(s)^k \mathcal{D}_{k,M}(1-s) + \mathcal{E}_k(s) \,,$$

where

$$\mathcal{D}_{k,N}(s) = \sum_{n=1}^{N} \frac{d_k(n)}{n^s} \,,$$

$MN = \left(\frac{t}{2\pi}\right)^k$, and $\chi(s) = \pi^{s-1/2}\Gamma(\frac{1-s}{2})/\Gamma(\frac{s}{2})$ is the factor from the functional equation for the zeta function, $\zeta(s) = \chi(s)\zeta(1-s)$. Taking $s = 1/2 + it$, we find that

$$\zeta(1/2+it)^k = \mathcal{D}_{k,N}(1/2+it) + \chi(1/2+it)^k \mathcal{D}_{k,M}(1/2-it) + \mathcal{E}_k(1/2+it) \,.$$

Assuming the error term is negligible, we obtain

$$\int_T^{2T} |\zeta(1/2+it)|^{2k}\, dt \sim \int_T^{2T} |\mathcal{D}_{k,N}(1/2+it)|^2\, dt + \int_T^{2T} |\mathcal{D}_{k,M}(1/2+it)|^2\, dt$$
$$+ 2Re \int_T^{2T} \chi(1/2-it)^k \mathcal{D}_{k,N}(1/2+it)\mathcal{D}_{k,M}(1/2+it)\, dt\ .$$

There is reason to believe that the cross term is smaller than the main term and that $MN = (t/2\pi)^k$ may be replaced by $MN = (T/2\pi)^k$. Thus, we expect that

$$\int_T^{2T} |\zeta(1/2+it)|^{2k}\, dt \sim \int_T^{2T} |\mathcal{D}_{k,N}(1/2+it)|^2\, dt + \int_T^{2T} |\mathcal{D}_{k,M}(1/2+it)|^2\, dt. \tag{9.1}$$

where $MN = (T/2\pi)^k$ and $M, N \geq 1$. We can prove this when $k = 1$ or $k = 2$, provided that M and N are both $\ll T$. When $k \geq 3$, the known bounds for $\mathcal{E}_k(s)$ are too large and it is difficult to show that the cross term really is small. (However, it might be possible to overcome these problems by appealing to a more complicated form of the approximate functional equation developed by A. Good [GD].) Our problem now is to determine an asymptotic estimate for the mean square of the Dirichlet polynomials $\mathcal{D}_{k,N}(1/2+it)$ and $\mathcal{D}_{k,M}(1/2+it)$.

Montgomery and Vaughan's mean value theorem, Theorem 7.2, gives

$$\int_T^{2T} \left|\mathcal{D}_{k,N}(1/2+it)\right|^2\, dt = \sum_{n\leq N} \frac{d_k(n)^2}{n}(T + O(n))\ .$$

By standard techniques one can show that

$$\sum_{n\leq N} d_k(n)^2 \sim \frac{a_k}{\Gamma(k^2)} N \log^{k^2-1} N$$

and that

$$\sum_{n\leq N} \frac{d_k(n)^2}{n} \sim \frac{a_k}{\Gamma(k^2+1)} \log^{k^2} N\ ,$$

where

$$a_k = \prod_p \left(\left(1-\frac{1}{p}\right)^{k^2} \sum_{r=0}^{\infty} \frac{d_k(p^r)^2}{p^r} \right)\ .$$

Thus, for $N \ll T$, we deduce that

$$\int_T^{2T} |\mathcal{D}_{k,N}(1/2+it)|^2\ dt \sim \frac{a_k}{\Gamma(k^2+1)} T \log^{k^2} N\,.$$

Using this with $M, N \ll T$ and $MN = (T/2\pi)^k$, we obtain the classical estimates for $I_1(T)$ and $I_2(T)$.

If $k \geq 3$, the condition $MN = (T/2\pi)^k$ forces at least one of M or N to be $\gg T$, so we need Theorem 7.3, the mean value for long Dirichlet polynomials. This requires good uniform estimates for the additive divisor sums

$$D_k(x,h) = \sum_{n \leq x} d_k(n) d_k(n+h)\,.$$

No such formula has been proved when $k > 2$, but a precise formula for the main term of $D_k(x,h)$ can be conjectured by a heuristic application of the circle method. This leads us to guess that

$$D_k(x,h) = m_k(x,h) + O(x^{1/2+\epsilon})$$

uniformly for $1 \leq h \leq x^{1/2}$, where $m_k(x,h)$ is a certain smooth function of x. Using this in Theorem 7.3, we obtain the

Conjecture 1. Let $N = (T/2\pi)^{1+\eta}$ with $0 \leq \eta \leq 1$. Then

$$\int_T^{2T} \left|\mathcal{D}_{k,N}(1/2+it)\right|^2 \, dt \sim w_k(\eta) \frac{a_k}{\Gamma(k^2+1)} T L^{k^2} \,,$$

where a_k is the product over primes defined previously and

$$w_k(\eta) = (1+\eta)^{k^2} \left(1 - \sum_{n=0}^{k^2-1} \binom{k^2}{n+1} \gamma_k(n) \left(1 - (1+\eta)^{-(n+1)}\right) \right) \,,$$

with

$$\gamma_k(n) = (-1)^n \sum_{i=0}^{k} \sum_{j=0}^{k} \binom{k}{i} \binom{k}{j} \binom{n-1}{i-1, j-1, n-i-j+1}$$

when $n \geq 1$ and$\gamma_k(0) = k$.

The conjecture restricts us to $N \ll T^2$. Thus, M and N in (9.1) must satisfy

$$M \ll T^2, \quad N \ll T^2, \quad \text{and} \quad MN = (T/2\pi)^k\,.$$

Writing $N = (T/2\pi)^{1+\eta}$ and $M = (T/2\pi)^{k-1-\eta}$, we find that

$$\int_T^{2T} |\zeta(1/2+it)|^{2k} \, dt \sim \int_T^{2T} |\mathcal{D}_{k,(T/2\pi)^{1+\eta}}(1/2+it)|^2 \, dt + \int_T^{2T} |\mathcal{D}_{k,(T/2\pi)^{k-1-\eta}}(1/2+it)|^2 \, dt\,,$$

with $0 \leq \eta \leq 1$. Hence,

$$\int_T^{2T} |\zeta(1/2+it)|^{2k} \, dt \sim \big(w_k(\eta) + w_k(k-2-\eta)\big) \frac{a_k}{\Gamma(k^2+1)} T L^{k^2}$$

Example:The 6th moment. Take $k = 3$. Then

$$\int_T^{2T} |\zeta(1/2+it)|^6 \, dt \sim \big(w_3(\eta) + w_3(1-\eta)\big)\frac{a_3}{\Gamma(10)} TL^9$$

for $0 \le \eta \le 1$. We find from the conjecture that

$$w_3(\eta) = 1 + 9\eta + 36\eta^2 + 84\eta^3 + 126\eta^4 - 630\eta^5 + 588\eta^6 + 180\eta^7 - 9\eta^8 + 2\eta^9 \,,$$

and one can verify that

$$w_3(\eta) + w_3(1-\eta) = 42$$

for $0 \le \eta \le 1$. Therefore

$$\int_T^{2T} |\zeta(1/2+it)|^6 \, dt \sim 42 \, \frac{a_3}{9!} TL^9 \,.$$

Example:The 8th moment. Take $k = 4$. Then

$$\int_T^{2T} |\zeta(1/2+it)|^8 \, dt \sim \big(w_4(\eta) + w_4(2-\eta)\big)\frac{a_4}{\Gamma(17)} TL^{16} \,,$$

where η and $2-\eta$ must be in $[0,1]$. This forces $\eta = 1$. Now

$$w_4(1) = 12012 \,.$$

Hence

$$\int_T^{2T} |\zeta(1/2+it)|^8 \, dt \sim 24024 \frac{a_4}{16!} TL^{16} \,.$$

Originally we thought we would be able to take $N > T^2$ in our formulas. In other words, we expected the error terms in

$$D_k(x,h) = m_k(x,h) + O_h(x^{1/2+\epsilon}) \,,$$

when used in conjunction with the long mean value theorem and averaged over h up to $x^{1-\epsilon}$, would cancel. We were surprised to find, however, that they accumulate once h exceeds $x^{1/2-\epsilon}$ and contribute to the main term. It would be very interesting to understand this behavior better.

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